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Generalization of the Euler-type solution to the wave equation

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Abstract
Generalization of the Euler-type solution to the wave equation is given. Peculiarities of the space–time structure of obtained waves are considered. For some particular cases interpretation of these waves as 'subliminal' and 'superluminal' is discussed. The possibility of description of electromagnetic waves by means of the scalar solutions is shown.

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The goal of this paper is to generalize the specific solution of the wave equation with four independent variables

$$\psi = (1/R)\, f\,(\Phi) \tag{1}$$

on the nonaxisymmetric case.

Here

$$R = \sqrt{(z-\beta\tau)^2 + (1-\beta^2)\rho^2} \qquad \Phi = \tau - \beta z \pm R \tag{2}$$

where the time variable is denoted as $\tau = ct$, c is the velocity of light (that is for scalar waves the wavefront velocity), ρ and z are the space–time variables in the cylindrical coordinates, f is an arbitrary function, and β is an arbitrary complex parameter. Using terms introduced by Courant [1], one can treat the wavefunction (1) as a family of relatively undistorted progressing waves with the distortion factor $1/R$ and the phase function Φ. The Euler spherical wavefunction $\psi = (1/r)\, f\,(\tau - r)\,, r = \sqrt{\rho^2 + z^2}$ and the axisymmetric waves of Brittingham type [2] $\psi = (1/(\tau - z))\, f\left(\tau + z - \rho^2/(\tau - z)\right)$ are degenerated cases of the above solution. Calculations of solution (1) have been based on the complex space–time ray theory [3]. However, result (1) may be obtained by means of linear transformation from the Euler wavefunction. Constructing nonaxisymmetric solutions, we use the known solution of the wave equation and linear transformations. The present research is connected with the theory of information transmission: in particular, with the problems of distortion of the signal shape [1] and directional wave formation.

It is known [4] that the homogeneous wave equation

$$\left(\frac{\partial^2}{\partial\tau^2} - \nabla^2\right)\psi = 0 \tag{3}$$

is satisfied by putting

$$\psi_{nm} = \mathrm{e}^{\mathrm{i}m\varphi} P_n^m(x) \cdot \frac{1}{r} v_n(r,\tau) \qquad x = \cos\theta. \tag{4}$$

Here ρ, θ, φ are the spherical coordinates, $P_n^m(x)$ is the associated Legendre function of the first kind, n and m are integers such that $n \geqslant m \geqslant 0$, and the factor $v_n(r,\tau)$ is a solution of the equation

$$\left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial r^2} + \frac{n(n+1)}{r^2}\right)v_n = 0.$$

The general solution of the above equation can be written in the form

$$v_n = r^{n+1}\left(\frac{\partial}{r\partial r}\right)^n \frac{f(\tau \mp r)}{r} \tag{5}$$

where $f(\tau \mp r)$ are arbitrary real functions. Collecting expressions (4) and (5) we represent solutions of the wave equation in terms of the transient spherical harmonics (see [5] for details). Note that the function $P_n^m(x)$, in general, may be replaced by some solutions of the Legendre differential equation $W_n^\mu(x)$, and $\mathrm{e}^{\mathrm{i}m\varphi}$ by $\mathrm{e}^{\mathrm{i}\mu\varphi}$ where μ is an arbitrary complex number. Generalization of the specific solution (1) is based on expressions (4) and (5).

Let us transform the variables z and τ into z_β and τ_β, where

$$z_\beta = \frac{(z - \beta\tau)}{\sqrt{1-\beta^2}} \qquad \tau_\beta = \frac{(\tau - \beta z)}{\sqrt{1-\beta^2}} \tag{6}$$

where β is an arbitrary complex parameter, and define the variables r_β, x_β, and φ_β by relations

$$r_\beta = \left[z_\beta^2 + \rho^2\right]^{1/2} \qquad x_\beta = z_\beta/r_\beta \qquad \varphi_\beta = \varphi. \tag{7}$$

Here and below the branches of square roots are arbitrarily fixed in general expressions, and the obvious relation

$$\rho^2/r_\beta^2 + x_\beta^2 = 1 \tag{8}$$

defines branch of root $(1 - x_\beta^2)^{1/2}$. On the ground that derivation of an analytical function is analogous to derivation of the real one, we transform the wave equation into

$$\left(\frac{\partial^2}{\partial\tau_\beta^2} - \frac{1}{r_\beta^2}\frac{\partial}{\partial r_\beta}\left(r_\beta^2\frac{\partial}{\partial r_\beta}\right) - \frac{1}{r_\beta^2}\frac{\partial}{\partial x_\beta}\left((1-x_\beta^2)\frac{\partial}{\partial x_\beta}\right) - \frac{1}{r_\beta^2(1-x_\beta^2)}\frac{\partial^2}{\partial\varphi^2}\right)\psi = 0 \tag{9}$$

and, recalling (4) and (5), write the solution of the above equation in the form

$$\psi_{n\mu} = \mathrm{e}^{\mathrm{i}\mu\varphi} W_n^\mu(x_\beta) r_\beta^n \left(\frac{\partial}{r_\beta \partial r_\beta}\right)^n \frac{f(\tau_\beta \mp r_\beta)}{r_\beta}. \tag{10}$$

Here we suppose that $f\left(\tau_\beta \mp r_\beta\right)$ is an analytical function.

In the case of the transformation

$$z_{\tilde\beta} = -\frac{(z - \tilde\beta\tau)}{\sqrt{\tilde\beta^2 - 1}} \qquad \tau_{\tilde\beta} = -\frac{(\tau - \tilde\beta z)}{\sqrt{\tilde\beta^2 - 1}} \tag{11}$$

we define the new variables $r_{\tilde\beta}, x_{\tilde\beta},\ \varphi_{\tilde\beta}$ as

$$r_{\tilde\beta} = \left[z_{\tilde\beta}^2 - \rho^2\right]^{1/2} \qquad x_{\tilde\beta} = z_{\tilde\beta}/r_{\tilde\beta} \qquad \varphi_{\tilde\beta} = \varphi \tag{12}$$

and by using the relation

$$x_{\tilde\beta}^2 - \frac{\rho^2}{r_{\tilde\beta}^2} = 1 \tag{13}$$

obtain the equation in the form (9), where $r_\beta,\ x_\beta$ and τ_β have to be replaced by $r_{\tilde\beta},\ x_{\tilde\beta}$ and $\tau_{\tilde\beta}$, and write its solution as

$$\psi_{n\mu} = \mathrm{e}^{\mathrm{i}\mu\varphi} W_n^\mu(x_{\tilde\beta}) r_{\tilde\beta}^n \left(\frac{\partial}{r_{\tilde\beta}\partial r_{\tilde\beta}}\right)^n \frac{f_\mp(\tau_{\tilde\beta} \mp r_{\tilde\beta})}{r_{\tilde\beta}}. \tag{14}$$

The above procedure can be treated as substitution of expressions (10) and (14) into the wave equation (3) represented in terms of variables (7) or (12).

We shall now derive families of wavefunctions from the solutions (10) and (14) by supposing that β and $\tilde\beta$ are real magnitudes and the sign '+' of square roots $\sqrt{1-\beta^2}$ and $\sqrt{\tilde\beta^2 - 1}$ is fixed. Here we use the notation β for values less than 1 and $\tilde\beta$ for values greater than 1. In this case the wavefunctions $\psi_{nm}(\beta)$ and $\psi_{nm}(\tilde\beta)$ permit the same interpretation.

(i) When $\beta \in [0, 1)$, the expression (6) is the Lorentz transformation of the variables $\tau, \bar z$, the variable x_β in (7) is the cosine of the polar angle $\theta_\beta \in [0, \pi]$ of the frame of reference moving with the velocity $v = \beta c$ in the direction of the $0z$ axis, and $W_n^\mu(x_\beta)$ is the associated Legendre function of the first kind $P_n^\mu(\cos\theta_\beta)$. Then we obtain from (10) solutions to the wave equation, which have the singularity at the moving point $\rho = 0$, $z = \beta\tau$. In the particular case $\mu = m$ we get spherical waves (the transient spherical harmonics) in the moving frame $\psi_{nm}(\beta)$, which may be treated as 'subluminal'.

(ii) When $\tilde\beta \in (1, \infty)$ it is convenient to use solution (14). It is easy to verify that the phase function $\Phi = \tau_{\tilde\beta} \mp r_{\tilde\beta}$ as well as the variables $r_{\tilde\beta}$ and $z_{\tilde\beta}$ are real and greater than zero inside a circular cone with the vertices moving with velocity greater than the velocity of light

$$z = -\rho\sqrt{\tilde\beta^2 - 1} + \tilde\beta\tau \tag{15}$$

and the variable $x_{\tilde\beta} \in (1, \infty)$. Then supposing $\mu = m$ we choose function $W_n^m(x_{\tilde\beta})$ in the form of the associated Legendre function of the first kind $P_n^m(x_{\tilde\beta}) = (x_{\tilde\beta}^2 - 1)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}x_{\tilde\beta}^m} P_n(x_{\tilde\beta})$. This function is regular and single-valued on the $x_{\tilde\beta}$ complex plane with the cut along the real axis from +1 to -1 [5]. When m is even, $P_n^m(x_{\tilde\beta})$ is the polynomial and when m is odd, the polynomial is multiplied by the algebraic function $(x_{\tilde\beta}^2 - 1)^{1/2}$. We fix sign '+' of square root since $r_{\tilde\beta} > 0$ and $\rho > 0$. As a result, replacing the factor $\exp(\mathrm{i}m\varphi)$ by $\sin(m\varphi)$ or $\cos(m\varphi)$, we get the real solution of the wave equation $\psi_{nm}(\tilde\beta)$ inside cone (15), having singularities at the point $\rho = 0$, $z = \tilde\beta\tau$ moving with the velocity greater than the velocity of light, as well as on the conical surface (15). This solution may be treated as the 'superluminal' one, though the fronts of wave perturbations move with the velocity of light.

It easy to verify that one can get from $\psi_{nm}(\beta)$ an expression analogous to that from $\psi_{nm}(\tilde\beta)$ by supposing $\beta > 1$ (and vice versa by $\tilde\beta < 1$ in $\psi_{nm}(\tilde\beta)$). In general, the constructed solutions (10) and (14) may be written in the form

$$\psi_{n\mu} = \psi(0)\,\mathrm{e}^{\mathrm{i}\mu\varphi} W_n^\mu(\tilde x)\, R^n \left(\frac{\partial}{R\partial R}\right)^n \frac{f(k(\tau - \beta z \mp R))}{R}. \tag{16}$$

Here R is defined by relation (2), $\tau_\beta\sqrt{1-\beta^2} = \tau_{\tilde\beta}\sqrt{\tilde\beta^2-1} = \tau - \beta z$, $\psi(0)$ and k are free complex constants, and variables x_β and $x_{\tilde\beta}$ are

$$\tilde{x} = \pm(z - \beta\tau)/R \tag{17}$$

where '+' corresponds to (7) and '−' to (11). Expression (16) is a generalization of axisymmetric solution (1), which is easy to see by choosing $n = \mu = 0$ and $k = 1$.

Expression (16) can be also represented in the form of the finite series

$$\psi_{n\mu} = \psi_0 e^{i\mu\varphi} W_n^\mu(\tilde{x}) \sum_{l=0}^{n} a_{nl}^{\mp} R^{-(l+1)} f^{(n-l)}(k(\tau - \beta z \mp R)) \tag{18}$$

where coefficients $a_{nl}^{\pm}$ depend on n, l only, and $f^{(n-l)}$ are the $(n-l)$th derivatives of the function f with respect to its argument. Note that by shift of z and τ, R and $\tilde{x}$ in (16) and (18) can be replaced by $R_{sh} = \sqrt{(z - \beta\tau + A)^2 + \rho^2(1-\beta^2)}$ and $\tilde{x}_{sh} = \pm(z - \beta\tau + A)/R_{sh}$, and the phase function Φ by $\Phi_{sh} = k(\tau - \beta z \mp R_{sh} + B)$, where A and B are free constants.

Let us now replace an arbitrary complex parameter β in the expression (16) by $1/\beta$. Then after simple transformations we write the family of solutions to the wave equation in the form

$$\psi_{n\mu} = \psi(0) e^{i\mu\varphi} W_n^\mu(x_T) T^n \left(\frac{\partial}{T\partial T}\right)^n \frac{f(k(z - \beta\tau \mp T))}{T} \tag{19}$$

where

$$T = \sqrt{(\tau - \beta z)^2 - (1-\beta^2)\rho^2} \qquad \text{and} \qquad x_T = \mp\frac{(\tau - \beta z)}{T}. \tag{20}$$

Hence, the axisymmetric solution is

$$\psi = \left(\frac{1}{T}\right) f(k(z - \beta\tau \mp T)) \qquad \text{here} \quad \psi(0) = 1 \tag{21}$$

and, finally, by supposing $\beta = 0$ we get

$$\psi = \left(\frac{1}{\sqrt{\tau^2 - \rho^2}}\right) f\left(z - \sqrt{\tau^2 - \rho^2}\right). \tag{22}$$

These expressions require an individual investigation. Here we select the following for special mention.

(i) The substitution $\beta \Rightarrow 1/\beta$ yields (19) from (16) and vice versa.
(ii) Supposing that β is real, $\tau - \beta z > 0$ and $k = 1/(2(\beta - 1))$, we get from (21) in the limiting case $\beta \to 1$ the axisymmetric solution to the wave equation of Brittingham type.
(iii) The distortion factor $1/\sqrt{\tau^2 - \rho^2}$ in (22) is a simple solution of the 2D wave equation, while the distortion factor of the Euler wavefunction $1/r$ is the simple solution of the 3D Laplace equation.

Note that so-called X-shape waves (see [6], table I, and also (20) and [7]) are particular solutions of the wave equation that one can get from (16), (19) by supposing $f = 1$, or from the general expression

$$\psi_{n\mu} = \psi(0) e^{i\mu\varphi} W_n^\mu(\bar{x}) \bar{R}^n \left(\frac{\partial}{\bar{R}\partial\bar{R}}\right)^n \frac{1}{\bar{R}} \tag{23}$$

where $\tilde{x}$, R and x_T, T are denoted as $\bar{x}$, $\bar{R}$.

This expression is a generalization of the simple solutions of the wave equation in the form of distortion factor $1/\bar{R}$ and does not, in general, describe progressing waves. Note that 'subliminal–superluminal' propagating waves of Euler type (1) were first discussed by Heaviside at the end of the 19th century [8]. Blokhintsev has derived the steady-state description of the propagating wave generated by supersound oscillator [9]. Later the axisymmetric solutions (1) were obtained in the problem of wave formation by a moving point source for $\beta \in (0, \infty)$ [10]. Formation of nonaxisymmetric waves by sources distributed on a superluminal circle was considered in [11]. The above solutions hold functions $f(\Phi)$. It is just a phase function $\Phi = k(\tau - \beta z - R)$ that defines velocity of fronts of wave perturbations (i.e. pulses or signals) equal to the velocity of light (sound) for 'superluminal' solutions of type (16), (19).

In conclusion, we discuss the peculiarities of application of the constructed solutions to description of the electromagnetic waves. We use SI units and denote the electric induction and magnetic field strength vectors as $\vec{D}$ and $\vec{H}$ and suppose that parameters β and $\tilde{\beta}$ are real magnitudes. As solutions of the wave equation (3), $\psi_{nm}(\beta)$ and $\psi_{nm}(\tilde{\beta})$, have singularities at the points moving along the $0z$ axis, it is natural to write them in cylindrical coordinates $\rho,\ \varphi,\ z$. Then Maxwell's equations are satisfied by the substitution that for TM waves is [12]

$$\begin{aligned} &D_\rho = \frac{\partial^2}{\partial\rho\partial z}u \qquad D_\varphi = \frac{1}{\rho}\frac{\partial^2}{\partial z\partial\varphi}u \qquad D_z = \left(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial\tau^2}\right)u \\ &H_\rho = \frac{c}{\rho}\frac{\partial^2}{\partial\varphi\partial\tau}u \qquad H_\varphi = -c\frac{\partial^2}{\partial\rho\partial\tau}u \qquad H_z = 0 \end{aligned} \tag{24}$$

and the scalar function $\psi = \partial u/\partial\tau$ is a solution of the wave equation. So, by using the wavefunctions $\psi_{nm}(\beta)$, $\psi_{nm}\left(\tilde{\beta}\right)$, and expression (24), we find at once the components of the magnetic field strength vector. To get the components of the electric induction vector, the integration of the above wavefunctions with respect to time should be performed. When β and $\tilde{\beta}$ are complex magnitudes, one can also get from (24) the components of vector $\vec{H}$.

In addition, the spherical wavefunctions written in the spherical coordinates of the moving frame $\psi_{nm}(\beta)$, $\beta \in (0, 1)$ for TM waves allow expression of the components of the electric and magnetic field vectors $\vec{D}_\beta$ and $\vec{H}_\beta$ in terms of scalar function $u_\beta(r_\beta, \tau_\beta)$ [12]

$$\begin{aligned} &D_{\beta r} = \left(\frac{\partial^2}{\partial r_\beta^2} - \frac{\partial^2}{\partial\tau_\beta^2}\right)u_\beta \qquad D_{\beta\theta} = \frac{1}{r_\beta}\frac{\partial^2}{\partial\theta_\beta\partial r_\beta}u_\beta \qquad D_{\beta\varphi} = \frac{1}{r_\beta\sin\theta_\beta}\frac{\partial^2}{\partial\varphi_\beta\partial r_\beta}u_\beta \\ &H_{\beta r} = 0 \qquad H_{\beta\theta} = \frac{c}{r_\beta\sin\theta_\beta}\frac{\partial^2}{\partial\varphi_\beta\partial\tau_\beta}u_\beta \qquad H_{\beta\varphi} = -\frac{c}{r_\beta}\frac{\partial^2}{\partial\theta_\beta\partial\tau_\beta}u_\beta. \end{aligned} \tag{25}$$

Here we have to use the relation $\partial u_\beta/\partial\tau_\beta = r_\beta\psi_{nm}(\beta)$.

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